6.5 Matrix Factorization

$$A = LU \qquad A_{n\times n} \text{ matrix.}$$

where

$$L = \begin{pmatrix} \times & & 0 \\ \times & \ddots & \\ \vdots & & \ddots \\ \times & \cdots & \times \ \times \end{pmatrix}, \qquad U = \begin{pmatrix} \times & \times & \cdots & \times \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \times \\ 0 & \cdots & 0 & \times \end{pmatrix}$$

Lower Triang. Upper triang.

If $A = LU$ and we want to solve the syst.

$$A\vec{x} = \vec{b} \iff LU\vec{x} = \vec{b}$$

We can proceed as follows:

① Solve $L\vec{y} = \vec{b}$; forward Subst. $O(n^2)$ operat.

② Solve $U\vec{x} = \vec{y}$; backward Subst. $O(n^2)$ opers.

Example If $A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{pmatrix}$

$\Rightarrow$

$A = LU.$

where $L = \begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 3/2 & 1 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 9/2 & 15/2 \\ 0 & 0 & -4 \end{pmatrix}$

If we want to solve

$$A\vec{x} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} = \vec{b}$$

① We solve first

$$L\vec{y} = \vec{b}$$

or

$$\begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 3/2 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$$

Forward Subst.

$y_1 = -1,$

$3/2\, y_1 + y_2 = 0 \Rightarrow y_2 = -\frac{3}{2}(-1) = \frac{3}{2}$

$3/2\, y_1 + y_2 + y_3 = 4 \Rightarrow$

$y_3 = 4 + \frac{3}{2} - \frac{3}{2} = 4.$

$$\therefore \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3/2 \\ 4 \end{pmatrix}$$

② then we solve: $U\vec{x} = \vec{y}$

$$\begin{pmatrix} 2 & -1 & 1 \\ 0 & 9/2 & 15/2 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3/2 \\ 4 \end{pmatrix}$$

Backward Subst.

Backward Subst.

$$-4x_3 = 4 \Rightarrow \boxed{x_3 = -1.}$$

$$\frac{9}{2}x_2 + \frac{15}{2}x_3 = 3/2$$

$$\Rightarrow \frac{9}{2}x_2 = \frac{3}{2} - \frac{15}{2}(-1) = \frac{18}{2} = 9 \Rightarrow x_2 = \frac{18}{9} = 2 \Rightarrow \boxed{x_2 = 2}$$

and

$$2x_1 - x_2 + x_3 = -1$$

$$\Rightarrow 2x_1 = -1 + x_2 - x_3 = -1 + 2 + 1 = 2 \Rightarrow \boxed{x_1 = 1}$$

Soln of linear Syst. $\tilde{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ ✓

Jump to 3'

Question: How can we factor a matrix A in the form: $A = LU$. ?

↳ Cheney-Kincaid. 6.5 (Burden)

6.4 LU FACTORIZATION.

We will illustrate the process with a 3x3 matrix.

Consider the linear system : $A\vec{x} = \vec{b}$

$$A\vec{x} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} = \vec{b}$$

Using Gaussian elimination (Naive)

1st step: $m_{21} = 3/2$, $m_{31} = 3/2$.

$$A^{(2)}\vec{x} = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 9/2 & 15/2 \\ 0 & 9/2 & 7/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3/2 \\ 11/2 \end{pmatrix} = \vec{b}^{(2)}$$

2nd step: $m_{32} = \dfrac{9/2}{9/2} = 1.$

$$A^{(3)}\vec{x} = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 9/2 & 15/2 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3/2 \\ 4 \end{pmatrix} = \vec{b}^{(3)}$$

Backward Substitution: $\vec{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

In general, if A is a 3×3 matrix and naive Gaussian elimination can be performed, after 2 steps A is transformed in an upper triangular matrix U. In fact,

If

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = A^{(1)} = \left(a_{ij}^{(1)}\right)$$

1st step: Elimination of entries below a_{11}. in 1st column.

$$m_{21} = \frac{a_{21}}{a_{11}}, \qquad m_{31} = \frac{a_{31}}{a_{11}}$$

$$A^{(2)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - m_{21}a_{11} & a_{22} - m_{21}a_{12} & a_{23} - m_{21}a_{13} \\ a_{31} - m_{31}a_{11} & a_{32} - m_{21}a_{12} & a_{33} - m_{31}a_{13} \end{pmatrix} = \left(a_{ij}^{(2)}\right)$$

$$= \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} \end{pmatrix}$$

2^{nd} step: Elimination of entry below a_{22} in 2^{nd} column.

$$m_{32} = \frac{a_{32}^{(2)}}{a_{22}^{(2)}}.$$

$$A^{(3)} = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ 0 & a_{22}^{(2)} & a_{32}^{(2)} \\ 0 & \overbrace{a_{32}^{(2)} - m_{32}a_{22}^{(2)}}^{0} & a_{33}^{(2)} - m_{32}a_{32}^{(2)} \end{pmatrix} = \left(a_{ij}^{(3)}\right) =$$

$$= \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ 0 & a_{22}^{(2)} & a_{32}^{(2)} \\ 0 & 0 & a_{33}^{(3)} \end{pmatrix} = U.$$

Define

$$M^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{pmatrix} \quad \text{and} \quad M^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{pmatrix}$$

It can be easily proved that

$$M^{(1)}A^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = A^{(2)} =$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - m_{21}a_{11} & a_{22} - m_{21}a_{12} & a_{23} - m_{21}a_{13} \\ a_{31} - m_{31}a_{11} & a_{32} - m_{21}a_{12} & a_{33} - m_{31}a_{13} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(3)} \end{pmatrix}$$

and also

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{pmatrix}}_{M^{(2)}} \underbrace{\begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(3)} \end{pmatrix}}_{A^{(2)}} = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ 0 & a_{22}^{(2)} & a_{32}^{(2)} \\ 0 & a_{32}^{(2)} - m_{32}a_{22}^{(2)} & a_{33}^{(2)} - m_{32}a_{32}^{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ 0 & a_{22}^{(2)} & a_{32}^{(2)} \\ 0 & 0 & a_{33}^{(3)} \end{pmatrix} = A^{(3)}$$

Therefore,

$$M^{(2)} A^{(2)} = A^{(3)} = U \quad (\text{upper triangular})$$

which is equivalent to

$$M^{(2)}\left(M^{(1)} A^{(1)}\right) = U$$

or

$$\left(M^{(2)} M^{(1)}\right) A = U$$

Then

$$\boxed{A = \left(M^{(2)} M^{(1)}\right)^{-1} U = \left(M^{(1)}\right)^{-1}\left(M^{(2)}\right)^{-1} U} \quad (5.1)$$

$M^{(2)}$ and $M^{(1)}$ are both invertible since

$$\det\left(M^{(2)}\right) = 1, \qquad \det\left(M^{(1)}\right) = 1.$$

Inverse of $M^{(2)}$ and $M^{(1)}$

$$\begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ m_{21}-m_{21} & 1 & 0 \\ m_{31}-m_{31} & 0 & 1 \end{pmatrix} = I.$$

$$L^{(1)} \qquad\qquad M^{(1)} \qquad = I.$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m_{32} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m_{32}-m_{32} & 1 \end{pmatrix} = I$$

$$L^{(2)} \qquad\qquad M^{(2)} \quad = I.$$

These $L^{(i)}$ matrices satisfy the property that their product is a lower triangular matrix L.

In fact,

$$L^{(1)}L^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix} = L$$

or $\qquad L^{(1)}L^{(2)} = L.$

Substitution in (5.1) leads to

$$A = \left(M^{(1)}\right)^{-1}\left(M^{(2)}\right)^{-1} U = L^{(1)}L^{(2)}U = LU$$

In conclusion,

$$A = LU, \quad \text{where } \begin{cases} L \text{ is lower triangular} \\ U \text{ is upper triangular.} \end{cases}$$

In general,

$$A^{(n)} = M^{(n-1)} M^{(n-2)} \cdots M^{(1)} A = U \quad (6)$$

or

$$A = \left(M^{(n-1)} \cdots M^{(1)}\right)^{-1} U = [M^{(1)}]^{-1} [M^{(2)}]^{-1} \cdots [M^{(n-1)}]^{-1} U$$

Defining $[M^{(i)}]^{-1} = L^{(i)}$

$$\boxed{A = \left(L^{(1)} \cdots L^{(n-1)}\right) U \equiv L U}$$

Key result.

where $L \equiv L^{(1)} \cdots L^{(n-1)}$.

Now

$$M^{(1)} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -m_{21} & 1 & 0 \cdots & 0 \\ -m_{31} & & \ddots & 0 \\ \vdots & 0 & & \ddots \\ -m_{n1} & & & 1 \end{pmatrix}$$

It can be easily proved that

$$L^{(1)} = [M^{(1)}]^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ m_{21} & 1 & 0 \cdots & 0 \\ m_{31} & & \ddots & 0 \\ \vdots & 0 & & \ddots \\ m_{n1} & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ +m_{21} & 1 & 0 \\ +m_{31} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} - m_{21} & 1 & 0 \\ m_{31} - m_{31} & 0 & 1 \end{pmatrix} = I.$$

and in general.

$$L^{(k)} = [M^{(k)}]^{-1} = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & & \\ & & 1 & & \\ & & m_{k+1,k} & \ddots & \\ & & m_{k+2,k} & & \\ & 0 & \vdots & & \\ & & m_{n,k} & & 1 \end{pmatrix}$$

$$M^{(1)}L^{(1)} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -m_{21} & 1 & 0 & \cdots & 0 \\ -m_{31} & 0 & \ddots & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ -m_{n1} & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ m_{21} & 1 & 0 & \cdots & 0 \\ m_{31} & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & 0 \\ m_{n1} & 0 & \cdots & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ -m_{21}+m_{21} & 1 & 0 & \cdots & & 0 \\ -m_{31}+m_{31} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ & \vdots & & & \ddots & 0 \\ -m_{n1}+m_{n1} & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} = I$$

and in general

$$M^{(k)}L^{(k)} = I$$

Now, if $\underline{\underline{n=3}}$

$$L^{(1)}L^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m_{32} & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix}$$

In general, for A $n \times n$ matrix nonsingular

$$L = L^{(1)}L^{(2)} \cdots L^{(n-1)} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ m_{21} & 1 & 0 & \cdots & 0 \\ m_{31} & m_{32} & 1 & 0 \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ \vdots & \vdots & & & 1 \\ m_{n1} & m_{n2} & \cdots & m_{n,n-1} & 1 \end{pmatrix}$$

We have been able to prove the following theorem:

Thm 6.18. – If G.E. can be performed on $A\vec{x}=\vec{b}$ without row interchanges, then the matrix A can be factored into the product $A = LU$.

where

$$U = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & & a_{2n}^{(2)} \\ \vdots & & & & \\ 0 & \cdots & & a_{n-1,n-1}^{(n-1)} & a_{n-1,n}^{(n-1)} \\ 0 & \cdots & \cdots & 0 & a_{nn}^{(n)} \end{pmatrix} = M^{(n-1)} \cdots M^{(1)} A.$$

$$L = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ m_{21} & 1 & 0 & \cdots & 0 \\ m_{31} & m_{32} & 1 & 0 \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ m_{n1} & m_{n2} & \cdots & m_{n,n-1} & 1 \end{pmatrix} = L^{(1)} L^{(2)} \cdots L^{(n-1)}$$

In our example,

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{pmatrix}, \qquad U = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 9/2 & 5/2 \\ 0 & 0 & -4 \end{pmatrix}$$

and $$L = \begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 3/2 & 1 & 1 \end{pmatrix}$$

It can be easily verified that

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 3/2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 0 & 9/2 & 15/2 \\ 0 & 0 & -4 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{pmatrix} = A$$

The generalization of this factorization to $n \times n$ matrices is straightforward.

In general, for 3×3 matrices

$$L = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ 0 & a_{22}^{(2)} & a_{32}^{(2)} \\ 0 & 0 & a_{33}^{(3)} \end{pmatrix}$$

Alternative procedure for LU decomposition.

ALGORITHM 6.4 (Based on this)

Example: 3x3

$$LU = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} =$$

\# of unknws = 6+6 = 12

$LU = A. \Rightarrow$

$$\begin{pmatrix} l_{11}u_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + l_{22}u_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33} \end{pmatrix} = (a_{ij})_{3\times 3}$$

knowns

Then we have 9 equs. and 12 unknws.

Need to specify 3 add. conditions

those are given over the diag. terms

$l_{ii} = 1, \quad \forall i$ Dolittle.

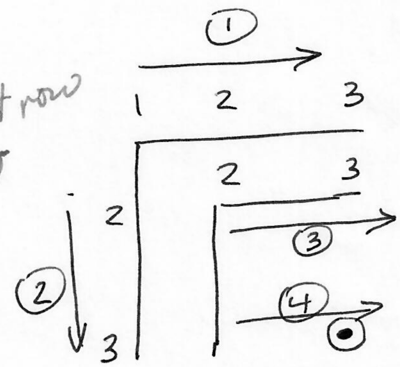
$u_{ii} = 1, \quad \forall i$ Crout.

$l_{ii} = u_{ii}, \quad \forall i$ Choleski.

Let's adopt Dolittle in our particular example

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{pmatrix} \qquad \begin{matrix} l_{11} = 1 \\ l_{22} = 1 \\ l_{33} = 1 \end{matrix}$$

$$\left.\begin{aligned} l_{11}\, u_{11} &= 2 \Rightarrow u_{11} = 2 \\ l_{11}\, u_{12} &= -1 \Rightarrow u_{12} = -1 \\ l_{11}\, u_{13} &= 1 \Rightarrow u_{13} = 1 \end{aligned}\right\} \text{First row } U$$



$$\left.\begin{aligned} l_{21}\, u_{11} &= 3 \Rightarrow l_{21} = \frac{3}{2} \\ l_{31}\, u_{11} &= 3 \Rightarrow l_{31} = \frac{3}{2} \end{aligned}\right\} \text{First column } L$$

2nd diag term $\quad l_{21} u_{12} + l_{22} u_{22} = 3 \Rightarrow u_{22} = \dfrac{3 - l_{21} u_{12}}{l_{22}} = \dfrac{3 + 3/2}{1} = 9/2$

2nd row U $\quad l_{21} u_{13} + l_{22} u_{23} = 9 \Rightarrow u_{23} = \dfrac{9 - l_{21} u_{13}}{l_{22}} = \dfrac{9 - 3/2}{1} = \dfrac{15}{2}$

2nd column L $\quad l_{31} u_{12} + l_{32} u_{22} = 3 \Rightarrow l_{32} = \dfrac{3 - l_{31} u_{12}}{u_{22}} = \dfrac{3 + 3/2}{9/2} = 1$

Lower right Corner Last entry.

$$l_{31} u_{13} + l_{32} u_{23} + l_{33} u_{33} = 5$$

$$\Rightarrow u_{33} = \frac{5 - l_{31} u_{13} - l_{32} u_{23}}{l_{33}} = 5 - 3/2 - \frac{15}{2} = -4$$

$$\therefore \quad L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ \frac{3}{2} & 1 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 9/2 & 15/2 \\ 0 & 0 & -4 \end{pmatrix}$$

Same as the one obtained by previous technique

Example of A nonsingular but row operations are needed for Gauss Elimination.

Gauss Elimination Process

$$A = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & -1 & 3 \\ 1 & 1 & 2 & 0 \end{pmatrix} \underset{E_1 \leftrightarrow E_3}{\sim} \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 4 & -4 \end{pmatrix} \sim$$

$$\overset{E_3 \leftrightarrow E_4}{\sim} \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

A cannot be factored as $A = LU$, because row interchanges are needed.

However, if the row interchanges were done from the beginning in A to produce a new matrix B, then the new matrix $B = LU$.

In our example two row interchanges were needed

$$\begin{matrix} E_1 \leftrightarrow E_3 \\ E_3 \leftrightarrow E_4 \end{matrix} \xrightarrow[\text{So new matrix}]{} B = \begin{pmatrix} \rightarrow E_3 \\ \rightarrow E_2 \\ \rightarrow E_4 \\ \rightarrow E_1 \end{pmatrix}$$

This matrix B can also be obtained from the matrix product $PA = B.$, where $P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

In fact,

$$B = PA = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & -1 & 3 \\ 1 & 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

Gauss Elimination on $PA = B$

$$B = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 3 & -3 \\ 0 & 1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$m_{21} = 0$
$m_{31} = 1$
$m_{41} = 0$

$m_{32} = -1$
$m_{42} = 1$

Therefore,

$$B = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

or $$PA = LU \Rightarrow A = (P^{-1}L)U$$

It can be shown that $P^{-1} = P^T$

Then,

Not lower triangular

$$A = (P^T L)U = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$